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TEN-DIMENSIONAL SUPERGRAVITY FROM LIGHT-LIKE INTEGRABILITY IN LOOP SUPERSPACE

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The idea of requiring integrability on light-like lines in superspace is generalised to loop superspace and used to give a complete derivation of all the constraints of on-shell ten-dimensional supergravity. These constraints can be interpreted as integrability conditions for certain suitably constrained string superfields.

1. Introduction

Light-like integrability is an important notion in understanding the geometry of self-dual Einstein [1] and self-dual Yang–Mills [2] theories. The basic idea is that the self-duality constraint is equivalent to the statement that the gauge field is pure gauge when restricted to any light-like (null) plane, i.e. on the null planes the covariant derivatives obey a flat algebra.

It turns out that the light-like integrability concept can explain the origin of the superspace constraints [3] for certain supersymmetric Yang–Mills theories [4,5]. In some cases these constraints put the theory on-shell in that they give rise to the field equations. In some other cases they only give the off-shell supermultiplet. A key ingredient in all this is that one considers supersymmetric light-like lines of one bosonic dimension and a certain number of fermionic dimensions*. In the context of supersymmetric Yang–Mills theory in ten dimensions, Witten showed some time ago that the relevant super light-like line is of dimension (1,8) and can be understood as the orbit of a superparticle described by a reparametrisation- and κ -invariant action in ten dimensions [5].

It is clearly desirable to understand also the supergravity equations in a similar way, i.e. as a consequence of light-like integrability conditions. Exploiting the

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* The terminology used here is slightly misleading. In a supersymmetric theory, a light-like “line” really means a light-like supersubmanifold with one bosonic dimension and a number of fermionic ones. We shall adopt the standard terminology, and refer to them as light-like lines.

κ -symmetry interpretation of the light-like lines, Witten showed that the on-shell ten-dimensional supergravity constraints are consistent with the constraints obtained by requiring the κ -invariance of a superparticle action formulated in ten-dimensional curved superspace [5]. However, the constraints obtained in this way are not, by themselves, sufficient to imply the ten-dimensional supergravity equations of motion. Furthermore, the relation between the constraints derived from the κ -symmetry of an action and those derived from the more direct notion of light-like integrability is not obvious. Therefore, it would be desirable to derive the ten-dimensional supergravity constraints directly from the considerations of light-like integrability. One such attempt was made by Chau and Milewski, who obtained the torsion and curvature superspace constraints which follow from the integrability conditions along light-like lines and studied the consequences of the Bianchi identities [6]. In contrast to the ten-dimensional supersymmetric Yang–Mills theory, it was found that they do not lead to the equations of motion of ten-dimensional supergravity. Instead, in order to obtain the complete set of equations of motion one needs to impose an additional constraint by hand.

One of the ingredients that went into the work of ref. [6] was to impose the light-like integrability conditions on the standard algebra of supercovariant derivatives corresponding to standard superspace. Some time ago a generalised flat superspace and a generalised set of supercovariant derivatives were introduced [7]. The distinguishing feature of the generalised superspace coordinates and derivatives is that all of them depend on an additional coordinate σ parametrising a circle. We shall call such a generalisation loop superspace. The fact that loop superspace is an S^1 -manifold (i.e. has an action of the circle group) allows the conversion of the two-form gauge field of ten-dimensional supergravity into a one-form on loop superspace. This one-form can be included in the definition of the supercovariant derivatives. The inclusion of the supergravity two-form in the formalism is natural from the point of view of string theory, since it couples to the string in much the same way that a Yang–Mills field couples to a particle.

The Kac–Moody algebra of ref. [7] has been rewritten in a supergeometric form which allows a natural generalisation to curved loop superspace [8]. It is the purpose of this paper to investigate the light-like integrability conditions with respect to the commutator algebra of the above-mentioned supercovariant derivatives in loop superspace. As we shall see, we can obtain the full set of superspace constraints that describe precisely on-shell ten-dimensional supergravity. In particular, the missing constraint in ref. [6], which had to be put in by hand, arises in our framework in a natural manner. This depends crucially on the fact that we work with a loop superspace extension of the standard superspace as it incorporates the two-form potential in a natural way. Finally, we show how the supergravity constraints can be interpreted as the integrability conditions for the existence of suitably constrained string fields. The constraints on such string fields can be solved explicitly.

2. Standard superspace

It is convenient first to give our notation and conventions for ordinary ten-dimensional superspace. Other dimensions can be treated in the same way if proper care is taken of the spinor properties in the specific dimension. Standard superspace in ten dimensions has coordinates $Z^M = (X^m, \theta^\mu)$, where X^m are the ten bosonic coordinates and θ^μ are the sixteen fermionic Majorana–Weyl spinorial coordinates. We use a notation where upper (lower) spinor indices are positive (negative) chiral. The standard supercovariant derivative is defined in curved superspace by

$$D_A = E_A^M \frac{\partial}{\partial Z^M} + \Omega_A + A_A, \quad (2.1)$$

where Ω_A is the Lorentz connection and A_A is a Yang–Mills connection. These supercovariant derivatives satisfy the commutation relations

$$\{D_A, D_B\} = -T_{AB}^C D_C + R_{AB} + F_{AB}, \quad (2.2)$$

where T_{AB}^C is the torsion, and R_{AB} and F_{AB} are the Lorentz and Yang–Mills curvatures*. Note that $M = (m, \mu)$ are coordinate indices while $A = (a, \alpha)$ are tangent space indices with respect to the preferred basis specified by the superviel-bein. The lorentzian connection Ω has the form

$$\Omega_A = \frac{1}{2} \Omega_A^{bc} L_{bc}, \quad (2.3)$$

where L_{bc} are the Lorentz generators.

The torsion and curvatures satisfy the Bianchi identities

$$\sum_{(ABC)} (D_A T_{BC}^D - R_{ABC}^D + T_{AB}^E T_{EC}^D) = 0, \quad (2.4)$$

$$\sum_{(ABC)} (D_A R_{BCD}^E + T_{AB}^F R_{FCD}^E) = 0, \quad (2.5)$$

$$\sum_{(ABC)} (D_A F_{BC} + T_{AB}^D F_{DC}) = 0, \quad (2.6)$$

where the sums are graded cyclic. In the rest of the paper we always assume that the tangent space group is lorentzian so $R_{ABc}^{\delta} = 0$ and $R_{AB\gamma}^{\delta} = \frac{1}{4}(\Gamma_c^{\delta})_{\gamma}^{\delta} R_{ABc}^{\delta}$.

In the framework described above, the torsion and curvature superfields contain too many component fields, and they do not correspond to any physical theory. It

* We use the superspace notation and conventions of ref. [9].

is therefore necessary to impose constraints. These constraints can be found knowing the supermultiplet structure of the supergravity theory to be described. For pure off-shell supergravity the multiplets are a $128 + 128$ multiplet containing the graviton and gravitino fields which couples to the conformal supercurrent introduced in ref. [10], an unconstrained scalar superfield of dimension -6 and a further unconstrained scalar superfield whose θ^0 -component is the physical dilaton [11]. The $128 + 128$ multiplet has the property that the curvature scalar of the graviton and the F -trace of the gravitino field equation vanish (at the linearised level); it is described in superspace by the constraints [12]

$$T_{\alpha\beta}{}^c = -i(\Gamma^c)_{\alpha\beta}, \quad T_{ab}{}^b = 0 \quad (2.7)$$

together with some conventional dimension-3/2 and dimension-1 constraints. When used in conjunction with the Bianchi identities these constraints imply

$$T_{\alpha\beta}{}^\gamma = T_{ab}{}^c = 0, \quad (2.8)$$

$$T_{\alpha\beta}{}^\gamma = \frac{1}{12}(\Gamma^{bc})_{\beta}{}^\gamma G_{abc} + \frac{1}{72}(\Gamma_{abcd})_{\beta}{}^\gamma G^{bcd}, \quad T_{ab}{}^c = 0, \quad (2.9)$$

where G_{abc} is a totally antisymmetric superfield. All the dimension 1, 3/2 and 2 torsions and curvatures can now be constructed as functions of G and its derivatives using eq. (2.4). One can then construct a superspace seven-form field strength whose purely vectorial component is the dual of G_{abc} . We note that the only non-conventional part of the first of the constraints in (2.9) sets the 1050-dimensional representation in $T_{\alpha\beta}{}^c$ to zero, and this is the constraint which must be relaxed to incorporate the dimension -6 auxiliary superfield [12, 13]. It turns out that in order to go on-shell (starting from the above constraints) it is necessary and sufficient to relate G_{abc} to the dilaton superfield \tilde{K} in the following way:

$$G_{abc} = \frac{3}{8}iD_\alpha(\Gamma_{abc})^{\alpha\beta}D_\beta e^{\tilde{K}}. \quad (2.10)$$

The θ^0 -component of G_{abc} now becomes a (supercovariantly) closed three-form, and in fact one can construct a superspace three-form $H = dB$. Clearly $dH = 0$, or, in index notation,

$$\sum_{(ABCD)} (D_A H_{BCD} + \frac{3}{2} T_{AB}{}^E H_{ECD}) = 0. \quad (2.11)$$

The components of H can be taken to be

$$H_{\alpha\beta\gamma} = 0, \quad H_{\alpha\beta c} = -i(\Gamma_c)_{\alpha\beta} e^{\tilde{K}}, \quad (2.12)$$

$$H_{ab\gamma} = -(\Gamma_{ab})_\gamma{}^\delta D_\delta e^{\tilde{K}}, \quad H_{abc} = G_{abc}. \quad (2.13)$$

These constraints are equivalent (by field redefinitions) to those given in refs. [5, 14].

Alternatively, again starting from the $128 + 128$ geometry (2.9)–(2.11) one can go on-shell by introducing a closed three-form H and imposing the constraints (2.12), from which (2.10) and (2.13) follow.

In order to compare directly with results we shall obtain in sect. 5 it is convenient to make some field redefinitions of the supervielbein and superconnection. These redefinitions can be expressed in terms of the basis one-forms $E^A \equiv dZ^M E_M^A$ and the connection one-form $\Omega \equiv E^A \Omega_A$. They take the form

$$\begin{aligned} E^a &\rightarrow e^{-\frac{1}{6}K} E^a, \\ E^a &\rightarrow e^{-\frac{1}{12}K} \left(E^a - \frac{1}{3} i E^a (\Gamma_a)^{\alpha\beta} D_\beta K \right), \\ \Omega &\rightarrow \Omega + \Sigma, \end{aligned} \quad (2.14)$$

where $K = -\frac{3}{2}\tilde{K}$, and

$$\Sigma_\alpha = -\frac{1}{12} (\Gamma^{bc})_\alpha{}^\beta D_\beta K L_{bc}. \quad (2.15)$$

In terms of these redefined variables the components of T and H up to dimension $\frac{1}{2}$ are

$$H_{\alpha\beta\gamma} = 0,$$

$$T_{\alpha\beta}{}^c = -i(\Gamma^c)_{\alpha\beta}, \quad H_{\alpha\beta c} = -i(\Gamma_c)_{\alpha\beta} e^K, \quad (2.16)$$

$$T_{\alpha\beta}{}^\gamma = H_{\alpha\beta c} = 0, \quad T_{ab}{}^c = -\frac{1}{2} (\Gamma_b \Gamma^c D K)_\alpha. \quad (2.17)$$

Flat superspace is obtained from the above geometry by setting $K = 0$. The flat covariant derivatives are, in standard coordinates,

$$D_a = \frac{\partial}{\partial X^a}, \quad D_\alpha = \frac{\partial}{\partial \theta^\alpha} + \frac{i}{2} (\Gamma^a \theta)_\alpha \frac{\partial}{\partial X^a}. \quad (2.18)$$

They obey

$$\{D_\alpha, D_\beta\} = i(\Gamma^a)_{\alpha\beta} D_a, \quad (2.19)$$

with all other commutators vanishing. The three-form H reduces to

$$H_{\alpha\beta c} = -i(\Gamma_c)_{\alpha\beta} \quad (2.20)$$

with all other components of H zero. The fact that this H satisfies the Bianchi identities (2.11) is due to the identity

$$(\Gamma^a)_{\alpha(\beta} (\Gamma_a)_{\gamma\delta)} = 0. \quad (2.21)$$

The derivation of the constraints given above is essentially group theoretic, but for many purposes it is useful to have a geometrical interpretation of them. Such an interpretation could lead to a new off-shell version of ten-dimensional supergravity, for example. The known off-shell version, briefly alluded to above, is not easy to work with because of the presence of negative-dimension fields, and is not obviously related to string theory because of the fact that it contains a six-form rather than a two-form gauge field. In the following it will be shown that the constraints of on-shell ten-dimensional supergravity do admit a geometric interpretation in terms of integrability along light-like lines.

3. Light-like integrability

In this section we shall review the concept of integrability along light-like lines in superspace [5]. First we shall show how it arises in the context of the supersymmetric particle and then apply it to the ten-dimensional super-Yang-Mills theory in order to derive the constraints for that theory.

The supersymmetric particle has (flat) superspace coordinates (X^a, θ^a) which are functions of time. As shown by Siegel [15], the superparticle action has a fermionic gauge symmetry which gauges away half of the components of the spinor coordinate θ^a . This is most clearly seen in the light-cone frame, where one can show that, for example, the projection $\theta_{(+\alpha)} \equiv (\Gamma^+ \theta)_\alpha$ of θ^a is a gauge degree of freedom. Combining this with time reparametrisations, which can gauge away, for example, the component X^- of X^a , one may say that the superparticle theory has a gauge orbit of dimension (1, 8), i.e. the subspace $(X^-, \theta_{(+\alpha)})$. After quantisation one can realise the generators of that gauge invariance as projections of the flat superspace covariant derivatives (2.18),

$$D^{(+\alpha)} = (\Gamma^+ D_{\text{flat}})^\alpha = \frac{\partial}{\partial \theta_{(+\alpha)}} - \frac{i}{2} (\Gamma^I \theta^+)^\alpha \frac{\partial}{\partial X^I} + \frac{i}{2} (\Gamma^+ \theta^-)^\alpha \frac{\partial}{\partial X^-}, \quad (3.1)$$

$$D^+ = \frac{\partial}{\partial X^-}.$$

An alternative way of looking at the above gauge invariance is to impose certain conditions on the wave function $\phi(X, \theta)$ of the superparticle using the generators (3.1), namely

$$D^{(+\alpha)} \phi = 0, \quad D^+ \phi = 0. \quad (3.2)$$

Notice that the integrability condition for these constraints to hold is that the derivatives in eq. (3.2) form a closed algebra. This is indeed the case, as follows from the flat superspace algebra (2.19),

$$\{D^{(+\alpha)}, D^{(+\beta)}\} = 2i(\Gamma^+)^{\alpha\beta} D^+, \quad [D^{(+\alpha)}, D^+] = 0. \quad (3.3)$$

The drawback in the discussion above is the lack of Lorentz covariance. The solution to this problem is to covariantise the light-cone, as suggested in ref. [16]. To this end one introduces new (auxiliary) variables (u_a^+, u_a^-, u_a^I) , $I = 1, \dots, 8$, which form an $\text{SO}(1, 9)$ matrix,

$$\begin{aligned} u^{+a} u_a^- &= 1, & u^{Ia} u_a^I &= -\delta^{IJ}, \\ u^{+a} u_a^+ &= u^{-a} u_a^- = u^{\pm a} u_a^I = 0. \end{aligned} \quad (3.4)$$

In particular, this means that the vectors $u^{\pm a}$ are light-like ones. Using these variables, one can *covariantly* project components of $\text{SO}(1, 9)$ vectors. For instance, one can define Γ -matrix projections

$$\Gamma^{\pm} = u^{\pm a} \Gamma_a, \quad (3.5)$$

which satisfy the relations

$$(\Gamma^+)^2 = (\Gamma^-)^2 = 0, \quad \Gamma^+ \Gamma^- + \Gamma^- \Gamma^+ = 2. \quad (3.6)$$

Note that $\frac{1}{2}\Gamma^+ \Gamma^-$ and $\frac{1}{2}\Gamma^- \Gamma^+$ form a complete set of projection operators. Their eigenstates are the $+$ and $-$ projections of spinors, for example,

$$\theta_{(\mp)\alpha} = (\Gamma^{\pm})_{\alpha\beta} \theta^{\beta}, \quad D^{(\pm)\alpha} = (\Gamma^{\pm})^{\alpha\beta} D_{\beta}. \quad (3.7)$$

Armed with those new tools, we can covariantly reformulate the statement of gauge invariance of the superparticle. Namely, we can say that the gauge orbit of the superparticle is defined by a light-like vector, u^{-a} , and is parametrised by $X^- = u^{-a} X_a$, $\theta^{(-\alpha)} = u^{-a} (\Gamma_a \theta)^{\alpha}$. Note that in the original presentation of Siegel the role of this light-like vector is played by the on-shell momentum p^a of the superparticle. According to the terminology of Witten [5], such orbits are called “light-like lines in superspace”. The covariantisation of the differential constraints (3.2) and the integrability conditions (3.3) for them is straightforward.

Our next step will be to apply the concept of light-like line integrability to the ten-dimensional supersymmetric Yang–Mills theory, as proposed in ref. [4]. This means that we shall demand that the $+$ projections of the covariant derivatives $\mathcal{D}_A = D_A + A_A(Z)$ satisfy integrability conditions, similar to (3.3). In other words, we shall require that the curvature two-form of the super-Yang–Mills theory should vanish when projected onto the dimension $(1, 8)$ light-like subspace, i.e. the gauge superfield connection becomes a pure gauge on light-like lines.

In detail, the above condition amounts to the following. We take the anticommutation relation (see eq. (2.2))

$$\{\mathcal{D}_{\alpha}, \mathcal{D}_{\beta}\} = i(\Gamma^a)_{\alpha\beta} \mathcal{D}_a + F_{\alpha\beta}, \quad (3.8)$$

and project it out with Γ^+ ,

$$\{\mathcal{D}^{(+\alpha)}, \mathcal{D}^{(+\beta)}\} = 2i(\Gamma^+)^{\alpha\beta} \mathcal{D}^+ + (\Gamma^+ F \Gamma^+)^{\alpha\beta}, \quad (3.9)$$

where $\mathcal{D}^{(+\alpha)}$ is defined in eq. (3.7), and $\mathcal{D}^+ \equiv u^{+a} \mathcal{D}_a$. Then we demand that the curvature term in eq. (3.9) vanishes,

$$(\Gamma^+ F \Gamma^+)^{\alpha\beta} = 0, \quad (3.10)$$

so that (3.9) would look like the flat relation (3.3). Note that the second integrability condition,

$$[\mathcal{D}^{(+\alpha)}, \mathcal{D}^+] = 0, \quad (3.11)$$

then follows from a Bianchi identity. The curvature tensor $F_{\alpha\beta}$ in eq. (3.8) is symmetric in α and β , so it can be decomposed into a $1-\Gamma$ and a $5-\Gamma$ term. Eq. (3.10) implies that both of these terms must vanish. For example, for the $5-\Gamma$ term we obtain

$$u_a^+ u_b^+ (\Gamma^a \Gamma^{c_1 \dots c_5} \Gamma^b)^{\alpha\beta} F_{c_1 \dots c_5} = 0. \quad (3.12)$$

Using the fact that u_a^+ and u_b^+ in eq. (3.12) are *arbitrary* light-like vectors, and with a little bit of Γ -matrix algebra, one can easily show that $F_{c_1 \dots c_5}$ is zero. Similarly it can be shown that the $1-\Gamma$ term is zero, and hence $F_{\alpha\beta} = 0$. This is the well-known constraint of on-shell ten-dimensional super-Yang–Mills theory.

The conclusion is that the requirement of light-like integrability in the gauge theory above unambiguously leads to the superspace constraints of the theory. One might pose the natural question: Does the same apply to ten-dimensional supergravity? This problem has been studied by Chau and Milewski, using eq. (2.2) with $F = 0$, who showed that most of the constraints of ten-dimensional supergravity could indeed be derived from light-like integrability, with one exception [6]. As we shall show in the following sections, this difficulty can be overcome by generalising the notion of superspace.

4. Loop superspace

Loop superspace, LM, is the space of maps from the circle S^1 to the superspace M. It can be coordinatised by $Z^M(\sigma) = (X^m(\sigma), \theta^\mu(\sigma))$, where σ ($0 \leq \sigma \leq 2\pi$) is the periodic circle coordinate. (A superstring can be described by its loop superspace coordinates which will be functions of a time-like parameter τ .) Given a tensor field on M we can construct in a straightforward way a corresponding “ultra-local” tensor field on LM, i.e. a tensor field whose components are local functions of $Z(\sigma)$, involving no derivatives with respect to σ , multiplied by

δ -functions. For example, if $X_{AB}(Z)$ is a rank-two tensor on M we can define a rank-two tensor field, $X_{AB}(\sigma, \sigma')[Z]$, on LM by

$$X_{AB}(\sigma, \sigma')[Z] = X_{AB}(Z(\sigma))\delta(\sigma - \sigma'). \quad (4.1)$$

Thus the structure equations for M (eqs. (2.2)), can be taken over into loop space with suitable minor modifications. However, these structure equations can be modified when M has a two-form gauge field B . This is done by exploiting the fact that LM is an S^1 -manifold. The action of the S^1 group is induced from translations mod 2π of the circle, $\sigma \mapsto \sigma + a \bmod 2\pi$; thus on LM , $Z(\sigma) \mapsto \bar{Z}(\sigma) = Z(\sigma + a)$. Let Π be the vector field on LM which generates these transformations,

$$\Pi^A(\sigma)[Z] = Z'^M(\sigma)E_M^A(Z(\sigma)), \quad Z'^M(\sigma) = \frac{d}{d\sigma}Z^M(\sigma). \quad (4.2)$$

We can now construct a one-form \hat{B} on LM by contracting Π with B , where the latter is now considered as an ultra-local two-form on LM . We thus define $\hat{B} = i_\Pi B$, where i_Π denotes the interior product operation with respect to the vector field Π . In index notation,

$$\begin{aligned} \hat{B}_A(\sigma)[Z] &= \int d\sigma' \Pi^B(\sigma') B_{BA}(Z(\sigma')) \delta(\sigma' - \sigma) \\ &= \Pi^B(\sigma) B_{BA}(Z(\sigma)). \end{aligned} \quad (4.3)$$

The gauge transformation of B is

$$\delta B = d\lambda, \quad (4.4)$$

where λ is a one-form parameter; this induces the following transformation of \hat{B} :

$$\delta \hat{B} = i_\Pi d\lambda = -d(i_\Pi \lambda), \quad (4.5)$$

where the last step follows from the identity $di_\Pi + i_\Pi d = \mathcal{L}_\Pi$, with \mathcal{L}_Π the Lie derivative along the vector field Π , and the fact that ultra-local fields are S^1 -invariant. Thus, for example, $\mathcal{L}_\Pi \lambda = 0$. In index notation,

$$i_\Pi \lambda = \int d\sigma \Pi^A(\sigma) \lambda_A(Z(\sigma)), \quad (4.6)$$

so

$$\delta \hat{B}_A(\sigma)[Z] = -D_A(\sigma) \int d\sigma' \Pi^B(\sigma') \lambda_B(Z(\sigma')), \quad (4.7)$$

where the covariant derivative in loop superspace is

$$D_A(\sigma) = E_A^M(Z(\sigma)) \frac{\delta}{\delta Z^M(\sigma)}. \quad (4.8)$$

This derivative can be extended to act on fields transforming nontrivially under Lorentz transformations and under (4.5) by

$$D_A(\sigma) = E_A^M \frac{\delta}{\delta Z^M(\sigma)} + \Omega_A(\sigma) + \Pi^B B_{BA}(\sigma). \quad (4.9)$$

The component version of this derivative was first given in flat superspace in ref. [7]. Subsequently a supergeometric description of this flat superspace derivative using loop superspace was given in the first paper of ref. [8]. Here we use an extension of this derivative to a curved superspace. A similar extension (including the Lorentz and Yang–Mills connection) was also proposed in the second paper of ref. [8]. In eqs. (4.8) and (4.9), $\delta/\delta Z^M$ is a functional derivative, i.e. $\delta Z^N(\sigma')/\delta Z^M(\sigma) = \delta_M^N \delta(\sigma - \sigma')$. The Lorentz connection Ω_A is defined as follows:

$$\Omega_A(Z(\sigma)) = \frac{1}{2} \Omega_A^{bc}(Z(\sigma)) L_{bc}(\sigma), \quad (4.10)$$

where $L_{ab}(\sigma)$ are the Lorentz generators satisfying the loop algebra

$$[L_{ab}(\sigma), L^{cd}(\sigma')] = 4\delta_{[a}^{[c} L_{b]}^{d]}(\sigma) \delta(\sigma - \sigma'). \quad (4.11)$$

These generators act on any representation of the Lorentz group in the usual way, supplemented by $\delta(\sigma - \sigma')$. The structure of $D_A(\sigma)$ is reminiscent of the structure of the momentum density $P_A(\sigma)$ for the Green–Schwarz superstring action, which also contains a term proportional to the superform B . The commutation relations obeyed by the derivatives (4.9) are straightforward to derive,

$$[D_A(\sigma), D_B(\sigma')] = (-T_{AB}^C D_C + R_{AB} + \Pi^C H_{CAB}) \delta(\sigma - \sigma'), \quad (4.12)$$

where H is the three-form field strength corresponding to B introduced in sect. 2.

The derivative $D_A(\sigma)$ acts on string wave functions, i.e. functionals of $Z(\sigma)$ which can transform under representations of the Lorentz group and which should also transform nontrivially under the transformation (4.5). For our purposes it will be sufficient to consider functionals $\Phi[Z]$ which are Lorentz scalars, but which transform by

$$\delta\Phi[Z] = i_\Pi \lambda \Phi[Z] \quad (4.13)$$

under the gauge transformation (4.5). Clearly

$$D\Phi = d\Phi = \hat{B}\Phi \quad (4.14)$$

defines a covariant exterior derivative D . In index notation the covariant derivative is just given by eq. (4.9) with the Lorentz connection omitted,

$$D_A(\sigma)\Phi = \left(E_A{}^M(Z) \frac{\delta}{\delta Z^M} + \Pi^C B_{CA}(Z) \right) (\sigma)\Phi[Z]. \quad (4.15)$$

It should be emphasised that the scalar functional $\Phi[Z(\sigma)]$ has no explicit σ -dependence (otherwise it would be a function on $LM \times S^1$ rather than on LM). On the other hand, the vielbein $E_M{}^A$, the Lorentz connection Ω_A and the two-form B_{AB} are ordinary functions of $Z(\sigma)$.

5. Light-like integrability in loop superspace

Our discussion of light-like integrability for ten-dimensional supergravity will follow the same general pattern as that for the Yang–Mills example described in sect. 3. The starting point here is the commutation relations given in eq. (4.12). A light-like line in curved superspace is one whose tangent vector at each point is a light-like vector in the sense described in sect. 3. A light-like line in loop superspace can be thought of as a loop of such lines. We then demand that projecting the commutation relations (4.12) onto such generalised light-like lines should give the following result:

$$\{D^{(+\alpha)}(\sigma), D^{(+\beta)}(\sigma')\} = 2i(\Gamma^+)^{\alpha\beta}(\sigma)(D^+(\sigma) - e^{K(Z(\sigma))}\Pi^+(\sigma))\delta(\sigma - \sigma'), \quad (5.1)$$

$$[D^+(\sigma), D^{(+\alpha)}(\sigma')] = 0, \quad (5.2)$$

where $D^{(+\alpha)}(\sigma) = (\Gamma^+)^{\alpha\beta}(\sigma)D_\beta(\sigma)$, $D^+(\sigma) = u^{+a}(\sigma)D_a(\sigma)$ and $\Gamma^+(\sigma) = u^{+a}(\sigma)\Gamma_a$. Note that the light-like vector u^{+a} now depends on σ . Comparing with eq. (3.3), we see that this resembles the flat-superspace algebra. In fact, it is almost the algebra of covariant derivatives in flat loop superspace as can be seen using eq. (2.20). The only difference is the presence of the scale factor e^K in eq. (5.1). If K were set to zero, the projected algebra would be exactly the flat one. However, it turns out that demanding precisely the flat algebra gives rise to constraints that are too restrictive, and so instead one must relax the requirements somewhat and allow the scale factor to multiply the flat-space expression for the three-form. It would be nice to find a natural geometrical interpretation for this factor; for now,

we shall simply take eqs. (5.1) and (5.2) as our definition of what we mean by light-like integrability. Now we shall work out the consequences of the first integrability condition (5.1). We have

$$T^{(+\alpha)(+\beta)c} = -2i(\Gamma^+)^{\alpha\beta} u^{+d} \delta_d^c, \quad (5.3)$$

$$H_c^{(+\alpha)(+\beta)} = -2i e^K (\Gamma^+)^{\alpha\beta} u_{+d} \delta_c^d. \quad (5.4)$$

The tensors in (5.3) and (5.4) can be decomposed into irreducible representations of the Lorentz group and, using the properties of the light-like vectors u_a^+ , we can derive the following results:

$$T_{\alpha\beta}{}^\gamma = 0, \quad T_{\alpha\beta}{}^c = -i(\Gamma^c)_{\alpha\beta}, \quad (5.5)$$

$$H_{\gamma\alpha\beta} = 0, \quad H_{c\alpha\beta} = -i e^K (\Gamma_c)_{\alpha\beta}. \quad (5.6)$$

Next we turn to the second integrability condition (5.2). Using eq. (5.1) in the Bianchi identity,

$$[\{D^{(+\alpha)}, D^{(+\beta)}\}, D^{(+\gamma)}] + \text{cyclic} = 0, \quad (5.7)$$

we find that eq. (5.2) implies

$$0 = [D^+, D^{(+\alpha)}] = e^K \Pi^B (T_E^{(+\alpha)+} - \delta_B^c u_c^+ D^{(+\alpha)} K). \quad (5.8)$$

This, together with the constraints (5.5) and (5.6), produces the following new constraints:

$$T_{ab}{}^c = -\frac{1}{2}(\Gamma_b \Gamma^c D)_a K, \quad H_{abc} = 0. \quad (5.9), (5.10)$$

The set of constraints (5.5), (5.6), (5.9) and (5.10) is the same as the set given in eqs. (2.16) and (2.17), and hence describes on-shell supergravity. It should be mentioned that eqs. (5.1) and (5.2) also lead directly to constraints on $T_{\alpha\beta}{}^\gamma$ and on parts of the curvature tensor. These quantities can also be derived from the T 's and the H 's given in eqs. (5.5), (5.6), (5.9) and (5.10) using the Bianchi identities, and it is straightforward to check that all the constraints which follow from eqs. (5.1) and (5.2) are indeed consistent with the Bianchi identities. (On this point we refer the interested reader to refs. [6, 17] where complete lists of the components of the torsion and curvature tensors are given.)

Thus we have shown that the full set of constraints for ten-dimensional supergravity can be derived from the principle of light-like integrability in loop superspace. As we mentioned at the end of sect. 3, one of these constraints did not follow from light-like integrability in the framework of standard superspace [6].

This is eq. (5.9). Now we can see the reason for that. In the new framework this constraint comes from eq. (5.8), which is proportional to Π^B . This term can only exist in loop superspace.

Another important issue we would like to discuss in this section is the meaning of the conditions (5.1) and (5.2) as integrability conditions in the proper sense of the word. In sect. 3 we mentioned that the commutation relations (3.3) allowed one to impose the restrictions (3.2) on the wave function of a superparticle. In fact, eqs. (3.3) were just *the integrability conditions* for eq. (3.2) to hold. It should also be noted that conditions (3.2) actually mean that the wave function does not depend on the superspace coordinates $X^-, \theta_{(+\alpha)}$ (they form the gauge orbit of the superparticle). Following that idea, it is tempting to interpret the supergravity constraints (5.1) and (5.2) as the integrability conditions for the following constraint on the superstring wave functional:

$$D^{(+\alpha)}\Phi[Z] = 0. \quad (5.11)$$

Note that the constraint (5.11) then implies a further restriction on $\Phi[Z]$ (when the geometry satisfies the light-like constraints),

$$D^+\Phi[Z] = e^K \Pi^+\Phi[Z]. \quad (5.12)$$

However, it turns out that the torsion constraints which follow from eqs. (5.11) and (5.12) are weaker than those which follow from eqs. (5.1) and (5.2) because derivatives of Φ appear on the right-hand side when we act on (5.11) or (5.12) with another covariant derivative and take the commutator. The light-like integrability constraints (5.1) and (5.2) are therefore sufficient, but not necessary, conditions for the existence of scalar superfields Φ satisfying (5.11) and (5.12).

The constraints (5.1) and (5.2) are thus of two types, “representation-preserving” and “conventional”, in the terminology of supergravity constraint analysis [18]. A subset of (5.1) and (5.2) is required in order that the representation defined by eqs. (5.11) and (5.12) should exist, and the remainder can be imposed by field redefinitions.

An important feature of the constraints (5.11) and (5.12) is that they can explicitly be solved. We shall demonstrate this first in the flat case. The commutation relations (5.1) and (5.2) are equivalent to

$$D_0^{(+\alpha)}\hat{B}^{(+\beta)} + D_0^{(+\beta)}\hat{B}^{(+\alpha)} - 2i(\Gamma^+)^{\alpha\beta}\hat{B}^+ = -2i(\Gamma^+)^{\alpha\beta}\Pi^+, \quad (5.13)$$

$$D_0^{(+\alpha)}\hat{B}^+ - D_0^+\hat{B}^{(+\alpha)} = 0, \quad (5.14)$$

where D_0 is the loop superspace version of the flat derivative (2.18). These equations are easily seen to be solved by

$$\hat{B}^{(+\alpha)} = D_0^{(+\alpha)}\Lambda[Z], \quad \hat{B}^+ = D_0^+\Lambda[Z] + \Pi^+, \quad (5.15), (5.16)$$

since $D_0^{(+\alpha)}\Pi^+ = 0$. In eqs. (5.15) and (5.16) Λ is a gauge parameter of the form $i_\Pi\lambda$. Hence it is possible to go to a gauge in which $\hat{B}^{(+\alpha)} = 0$, in which case eq. (5.11) reduces to

$$D_0^{(+\alpha)}\Phi = 0, \quad (5.17)$$

and this is easily seen to be solved by

$$\Phi[X^a, \theta^\alpha, u] = \Psi[X^+, \tilde{X}^I, \theta_{(-\alpha)}, u], \quad (5.18)$$

where

$$\tilde{X}^I = X^I - \frac{1}{2}i\theta^+\Gamma^I\theta^- \quad (5.19)$$

and $\Psi[X^+, \tilde{X}^I, \theta_{(-\alpha)}, u]$ is an arbitrary functional of its arguments. In other words, the constraint (5.11) can be interpreted as an *analyticity condition* on the wave functional. Its solution is an analytic functional, which does not depend on the variables $X^-, \theta_{(-\alpha)}$. We emphasize that the functionals in eq. (5.18) must also depend on the variables $u^{\pm a}, u^{Ia}$, in order to maintain manifest Lorentz invariance. In other words, the $SO(1,9)$ variables u should be viewed as harmonic variables [16, 19]. In a general gauge the solution to eq. (5.11) will then have the form

$$\Phi = e^\Lambda \Psi, \quad (5.20)$$

where Λ is a gauge parameter.

We can generalise the above argument to the curved case as well. The same integrability conditions make sure that one can find a special gauge transformation

$$\Phi \rightarrow \exp\left(\int d\sigma \Pi^A \lambda_A\right) \Phi, \quad (5.21)$$

and a special coordinate shift

$$Z^M \rightarrow Z^M(Z, u), \quad (5.22)$$

such that the vielbein and Π terms in $D^{(+\alpha)}$ (as well as D^+) can be gauged away, and those derivatives become partial ones. Then the analyticity conditions (5.1) and (5.2) can be explicitly solved in terms of an arbitrary functional of $X^+, X^I, \theta_{(+\alpha)}$. The implications of this fact could be rather far reaching, as experience with other supersymmetric theories in the harmonic superspace approach [19, 20] has shown.

6. Comments

It is well known that requiring κ -invariance of a standard Green–Schwarz superstring action also leads to superspace constraints in ten dimensions [5,21]. Furthermore, the constraints generated this way are equivalent to the constraints of on-shell ten-dimensional supergravity. To see this we note that the action of ref. [21] contains a scalar field k which may be absorbed into the supervielbein as the Wess–Zumino term is independent of the latter. Hence it is consistent to set $k = 0$ in the constraint equations derived from the requirement of κ -symmetry in ref. [21]. When one does this and imposes allowable conventional constraints one recovers the equations describing on-shell supergravity. (By allowable conventional constraints we mean those that preserve the constraints imposed by κ -symmetry.) Thus light-like integrability and κ -invariance lead to equivalent superspace constraints, as is also true for a superparticle in a background (abelian) Yang–Mills field [5]. There is a difference, however, in that light-like integrability leads directly to the equations of motion whereas the constraints following from κ -symmetry need to be supplemented with conventional constraints, in the string case.

It would be of interest to investigate the significance of the fact that light-like integrability is compatible with the existence of analytic string fields satisfying eqs. (5.11) and (5.12). It may be that such superfields could be of use in constructing superstring theories.

Further directions that seem worth pursuing are as follows. Here we have considered only $N = 1$, $d = 10$ supergravity without Yang–Mills fields. It is an important open problem to include the coupling to a supersymmetric Yang–Mills system. This would be straightforward for a conventional coupling, but the (anomaly-free) coupling of interest involves Chern–Simons terms. Also of interest is a derivation of the $N = 2$, $d = 10$ supergravity constraints from the light-like integrability for the loop-superspace algebra given in ref. [22]. Finally we note that the Kac–Moody-type loop superspace algebras for strings have a natural generalization to all super p -branes [8]. In particular, a Kac–Moody type algebra exists for the eleven-dimensional supermembrane. It would be very interesting to apply the ideas of light-like integrability to that case. In this connection we note that the generalisation of loop superspace is the space of maps from Σ to eleven-dimensional superspace M , where Σ is a two-dimensional manifold. This space, ΣM , say, is no longer an S^1 manifold, but instead becomes equipped with a special (non-local) bi-vector field, $\Pi^{AB}(\sigma, \sigma')[Z]$ [8]. This bi-vector field can be used in much the same way as the vector field $\Pi^A(\sigma)[Z]$ in loop superspace. For example the three-form gauge potential of eleven-dimensional supergravity can be converted into a one-form on ΣM using it.

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